Information processing by a controlled coupling process

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Abstract

This Letter proposes a controlled coupling process for information processing. The net effect of conventional coupling is isolated from the dynamical system and is analyzed in depth. The stability of the process is studied. We show that the proposed process can locally minimize the smoothness and the fidelity of dynamical data. A digital filter expression of the controlled coupling process is derived and the connection is made to the Hanning filter. The utility and robustness of proposed approach is demonstrated by both the restoration of the contaminated solution of the nonlinear Schrödinger equation and the estimation of the trend of a time series.

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The topic of synchronization and chaos control has attracted much attention in the past decade [1–9]. Active research in this area has contributed greatly to the understanding of a wide class of complex phenomena, including synchronization in secure communication [2], electronic circuits [3], and nonlinear optics [4], coherence transfer in magnetic resonance [5], and oscillation in chemical and biological systems [6]. In fact, the study of this topic leads to many practical applications in the aforementioned fields and a realistic scheme for shock capturing [7] in fluid dynamics. An interesting signal processing scheme was proposed by Lindner *et. al.* [8] in the context of synchronization. It is beneficial to explore further the potential of synchronization and chaos control techniques to digital signal pro-

cessing (DSP) and information autoregression (IAR). Both DSP and IAR are of crucial importance to telecommunication, biomedical imaging, seismic, radar, pattern recognition, missile guidance, target tracking, autonomous control, and a wide variety of other signals and data processing for commerce, finance, defense and scientific interests. Despite of the great achievements of traditional DSP and IAR techniques, many real-world problems remain unsolved due to the nonlinearities, non-Gaussian noises and limited real-time working conditions often encountered in applications. Solving these complex problems would require the development of innovative, flexible, fast, nonlinear DSP and IAR algorithms. The objectives of this Letter is to develop a synchronization-based, realistic technique for DSP and IAR.

We consider a nonlinear dynamical system consisting of N identical subsystems, which are coupled via the nearest and next-to-the-nearest neighborhood sites

$$
\begin{aligned}
\frac{du_j}{dt} &= f(u_j) + N_j(t) \\
&+ a(u_{j+2} - u_{j+1}) + (b - 3a)(u_{j+1} - u_j) \\
&+ a(u_{j-2} - u_{j-1}) + (b - 3a)(u_{j-1} - u_j),
\end{aligned}
\tag{1}
$$

where $u_j \in [0, \infty) \times R^n$, f is a nonlinear function of u_j which might undergo chaotic dynamics, $N_j(t)$ is the noise, a and b are scalar hyper-diffusive and diffusive coupling parameters, respectively. Equation (1) is a generalization of that given by Lindner $et.$ $al.$ [8]. The coupling scheme in Eq. (1) is strongly dissipative and a synchronous state can be attained for the chaotic Duffing oscillators, $\dot{u} = (\dot{x}, \dot{y}) = (y, -0.3y - x^3 + 11 \cos t)$, by using appropriate coupling parameters [9]. It is important to analyze in depth the effect of the coupling in Eq. (1). To this end, we introduce the following controlled coupling process

$$
\begin{aligned}
\frac{du_j}{dt} &= \theta_{t_1, t_2, \cdots, t_k}(t) \left[f(u_j) + N_j(t) \right] \\
&+ \bar{\theta}_{t_1, t_2, \cdots, t_k}(t) \left[a(u_{j+2} - u_{j+1}) + (b - 3a)(u_{j+1} - u_j) \right. \\
&+ \left. a(u_{j-2} - u_{j-1}) + (b - 3a)(u_{j-1} - u_j) \right],
\end{aligned}
\tag{2}
$$

where $\theta_{t_1, t_2, \cdots, t_k}(t)$ is a control function which consists of a train of Heaviside type intervals

and $\bar{\theta}_{t_1,t_2,\cdots,t_k}(t)$ is the complement of $\theta_{t_1,t_2,\cdots,t_k}(t)$ in the domain $[0,\infty)$ [i.e., $\bar{\theta}_{t_1,t_2,\cdots,t_k}(t) = 1 - \theta_{t_1,t_2,\cdots,t_k}(t)$]. Both control functions are depicted in FIG. 1. During the time interval $0 \leq t \leq t_1$, each subsystem, including possible noise, evolves freely. At time t_1, the coupling is switched on and the time evolution of the system is entirely governed by the controlled coupling until t_2. The subsystems return to the state of free evolution in the next time interval.

The controlled coupling process, Eq. (2), allows a detailed analysis of the *net effect* of the coupling in Eq. (1). At time $t_1 \leq t \leq t_2$, a discrete form of Eq. (2) can be given by

$$
\begin{aligned}
u_j^{S+1} &= u_j^S + R(u_{j-1}^S - 2u_j^S + u_{j+1}^S) \\
&\quad + T(u_{j-2}^S - 4u_{j-1}^S + 6u_j^S - 4u_{j+1}^S + u_{j+2}^S), \\
u_j^0 &= u_j(t_1) \qquad j = 1, \ldots, N,
\end{aligned}
\tag{3}
$$

where $R = b\Delta t$, $T = a\Delta t$ and iteration parameter S are user-specified constants. The controlled coupling process (3) is conditionally stable. Neglecting the boundary modifications, we can rewrite Eq. (3) in a matrix form,

$$
U^{S+1} = AU^S,
\tag{4}
$$

where $U^S = (u_1^S, u_2^S, \ldots, u_N^S)^{\mathrm{T}}$, and the banded matrix A has nonzero coefficients: $a_{j,j-2} = a_{j,j+2} = T$, $a_{j,j-1} = a_{j,j+1} = R - 4T$, and $a_{j,j} = 1 - 2R + 6T$, for $j = 1, 2, \ldots, N$. If all of the eigenvalues of A are smaller than unity, the iterative correction $\epsilon^{S+1} = ||U^{S+1} - U^S||$ will decay, then the process is stable. Since each diagonal term of the matrix is a constant, the eigenvectors of A can be represented in terms of a complex exponential form,

$$
U_j^S = q^S e^{i\gamma j},
\tag{5}
$$

where $i = \sqrt{-1}$ and γ is a wavenumber that can be chosen arbitrarily. Substituting Eq. (5) into Eq. (4) and removing the common term $e^{i\gamma j}$, we obtain an explicit expression for the eigenvalue q:

$$
q = 1 + 2T(2\cos^2\gamma - \cos\gamma - 1) + 2(R - 3T)(\cos\gamma - 1).
\tag{6}
$$

For a stable process, the magnitude of this quantity is required to be smaller than unity,

$$q^2 < 1. \tag{7}$$

For the case of $T = 0$, q is the maximum when $\cos \gamma = -1$. Thus, the controlled coupling process is stable provided $0 < R < \frac{1}{2}$. On the other hand, if $R = 0$, our analysis indicates that $0 > T > -\frac{1}{2} \frac{1}{(\cos^2 \gamma - 2 \cos \gamma + 1)}$, which also takes an extremum at $\cos \gamma = -1$. Therefore, the controlled coupling process is stable provided $0 > T > -\frac{1}{8}$. Under these conditions, we have $\epsilon^{S+1} \leq \epsilon^S$, for any $S \in \mathbb{Z}^+$.

It is interesting to noted that the controlled coupling process, Eq. (2), provides not only a treatment of nonlinear dynamical systems, but also a powerful, realistic approach to DSP and IAR of real-world problems. For IAR, Whittaker [10] suggested a method of trend estimation by the global minimization of fidelity and smoothness. The latter is defined by the accumulated power of the finite difference. Hodrick and Prescott [11] provided a concrete version of Whittaker's approach. Recently, Mosheiov and Raveh [12] proposed a linear programming approach to estimate the trend by employing the sum of the *absolute* values rather than the common sum of squares to measure the smoothness and fidelity. In the present approach, the terms in the first and second brackets of Eq. (3) are the second order and fourth order pointwise measures of smoothness, which are denoted as $\Delta^2 u_j^S$ and $\Delta^4 u_j^S$, respectively. To have a better understanding of this controlled coupling process, we rewrite Eq. (3) as

$$[u_j(t_1) - u_j^S] + [Rv_j^{S-1} + Tw_j^{S-1}] = 0, \quad j = 1, \ldots, N, \tag{8}$$

where $v_j^{S-1} = \sum_{k=0}^{S-1} \Delta^2 u_j^k$, and $w_j^{S-1} = \sum_{k=0}^{S-1} \Delta^4 u_j^k$. It is clear that the expression in the first square bracket is the local measure of the fidelity, while the expression in the second square bracket is the accumulative local measures of smoothness. Due to $\epsilon^{S+1} = ||U^{S+1} - U^S|| = ||R\Delta^2 U^S + T\Delta^4 U^S||$, is actually a global smoothness measure of estimated trend at the Sth iteration. One can argue that as the iterative process is carried out for a longer time, the estimated trend becomes smoother, while the deviation of U^S from

$U^0 = U(t_1) = [u_1(t_1), u_2(t_1), \ldots, u_N(t_1)]^\mathrm{T}$ becomes larger. At each step of the iteration, this process guarantees that the sum of the local deviation from $u_j(t_1)$ and the accumulative local measure of smoothness equals to zero. As such, the result of each iteration is optimal in the sense of minimization, for the given input and the set of parameters R and T. Two smoothing parameters R and T, and the iteration parameter S, govern the fundamental tradeoff between the smoothness and fidelity. In practice, R and/or T can be pre-fixed and only the iteration parameter S is optimized to achieve desired results. In comparison, the previous IAR methods seek for global minimizations over the entire domain to obtain optimal estimates, while the present controlled coupling process forces the sum of smoothness and fidelity to pass through zero at each iteration to give an optimal trend. The advantage of the proposed controlled coupling is its simplicity, robustness and efficiency.

For DSP, it is important to analyze the relationship between the proposed process and digital filters. To this end, we explore a weighted average representation of the controlled coupling process. For simplicity, we consider the case of $T = 0$. We first set S to 1, then the controlled coupling process gives

$$u_j^1 = Ru_{j-1}(t_1) + (1 - 2R)u_j(t_1) + Ru_{j+1}(t_1), \quad j = 1, \ldots, N, \tag{9}$$

which is clearly a local weighted average form for $u_j(t_1)$. In general, after S iterations, the controlled coupling process can be represented as:

$$u_j^S = \sum_{k=j-S}^{j+S} W(k, S)u_k(t_1), \tag{10}$$

where weight function $W(k, S)$ has the general form of

$$W(k, S) = \begin{cases} \sum_{h=0}^{(S-k)/2} & g(k, S, 2h) \quad\quad \text{when } S - k \text{ even} \\ \sum_{h=1}^{(S-k+1)/2} & g(k, S, 2h - 1) \quad \text{when } S - k \text{ odd,} \end{cases} \tag{11}$$

and

$$g(k, S, h) = \frac{S! R^{S-h} (1 - 2R)^h}{(\frac{S+k-h}{2})! (\frac{S-k-h}{2})! h!}. \tag{12}$$

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It can be easily verified that,

$$\sum_{k=j-S}^{j+S} W(k, S) = 1, \tag{13}$$

and

$$W(-k, S) = W(k, S) \quad \forall k = 1, \ldots, S. \tag{14}$$

Equation (10) indicates that the controlled coupling process can be viewed as a kernel smoother for IAR and a low-pass filter for DSP. The implementation of the controlled coupling process becomes extremely simple due to the existence of Eq. (10). Therefore, the weighted average form (10) is numerically very useful. The weights assignment of the controlled coupling process filter is analogous to that of other kernel regression methods. For a reasonable choice of R and S, the greater or smaller weight will be assigned to the points close or far away from $u_j(t_1)$, respectively, see Table I and FIG. 2. Obviously, the distribution of the weights has a Gaussian shape when S is sufficiently large.

A simple moving average filter can be constructed by convolving the mask $\left(\frac{1}{2}, \frac{1}{2}\right)$ with itself $2S$ times. when $S = 1$, such a filter is the Hanning filter [13] $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$. In our case, if we set $R = \frac{1}{4}$ in the Eq. (9), the present controlled coupling process has identical filter coefficients as those of the Hanning filter. Thus, the proposed controlled coupling process filter can be viewed as a generalization of the Hanning filter. A similar analysis including a non-vanishing T can be carried out.

In the rest of this Letter, we demonstrate the utility of the proposed approach through numerical experiments. First, we consider the signal extraction from noisy data by the controlled coupling process (2). The underlying nonlinear dynamic system is chosen as the nonlinear Schrödinger equation [14]

$$i\frac{\partial \Psi}{\partial t} + \frac{\partial^2 \Psi}{\partial x^2} + 2|\Psi|^2\Psi = 0, \quad x \in [0, L], \tag{15}$$

with periodic boundary conditions $\Psi(x + L, t) = \Psi(x, t)$ and the period $L = 2\sqrt{2}\pi$. This system is computationally difficult due to possible numerically induced chaos [14]. In this

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study, Eq. (15) is allowed to evolves freely with the initial condition of the form $\Psi(x,0) = 0.5 + 0.05 \cos\left(\frac{2\pi}{L}x\right) + i10^{-5} \sin\left(\frac{2\pi}{L}x\right)$. At $t = t_1 = 8.0$, the system is perturbed and its solution is contaminated by the Gaussian white noise to a signal-to-noise ratio (SNR) of 39.25 dB, see FIG. 3. The controlled coupling process is used to restore the solution from noisy dynamical data $U(t_1)$ during the time period $t_1 \leq t \leq t_2$. The parameters (R, T, S) are chosen as $(0.25, -0.05, 10)$. From FIG. 3, it is clear that the unwanted noise is satisfactorily suppressed by the proposed process. The restored solution matches well with the noise-free solution and its SNR is as high as 50.63 dB.

We next consider the trend estimation of a benchmark time series, the 'Sales of Company X' series [15,12]. Such a time series can be regarded as $U(t_1)$, produced by an unknown and unpredictable dynamic system, and its analysis is of practical importance. The Sales is a monthly series ranging from January 1965 to May 1971 and has a monotonic growing trend and a clearly identifiable seasonal component. Guided by the earlier stability analysis, we choose three sets of (R, T) values, $(0.4, 0)$, $(0, -0.12)$, and $(0.25, -0.05)$, for which nearly optimal results are obtained at $S = 38, 700$ and 52, respectively, see FIG. 4. The Neumann boundary condition is used to in this case. As shown in FIG. 4, trends estimated by using three sets of parameters are almost identical and provide similar long-run tendency. An important feature is that, the slope of the trend undergoes a clear change around the 28th month, which agrees with the finding in Ref. [12].

In conclusion, we introduce a controlled coupling process for the synchronization of spatiotemporal systems. The proposed process isolates the conventional coupling from the dynamical system and provides an in-depth analysis of the coupling effect. Numerical stability of the proposed process is analyzed. In the context of trend estimation, a comparison is given to several standard nonparametric methods [10–12], which globally minimization the smoothness and fidelity. The proposed process is shown to balance these features in each step of time evolution, without resorting to the minimization process, and thus, is numerically simpler than the existing methods. In the context of signal processing, a digital filter expression of the controlled coupling process is derived and the connection of the proposed

approach to the standard Hanning filter [13] is made. The proposed process is applied to signal restoration and trend estimation.

In the numerical experiment of signal restoration, the solution of the nonlinear Schrödinger equation is contaminated by noise at the end of the first time period t_1. The controlled coupling process is utilized to restore the waveform. We show that the unwanted noise can be effectively removed by 10 iterations. In the other experiment, a real-world time series generated by some unknown dynamical process is studied. The objective is to estimate the trend of the time series. Excellent trend estimations are obtained by using three different sets of parameters. The numerical results are in good agreement with those in the literature [12] and the present approach is simpler. Obviously, the proposed approach can be easily generalized to two spatial dimensions for image processing and the present investigation opens up a new opportunity to develop other synchronization-based, realistic information process methods.

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TABLE I. The filter weights $[W(k,6)]$ of the controlled coupling process ($T = 0$).

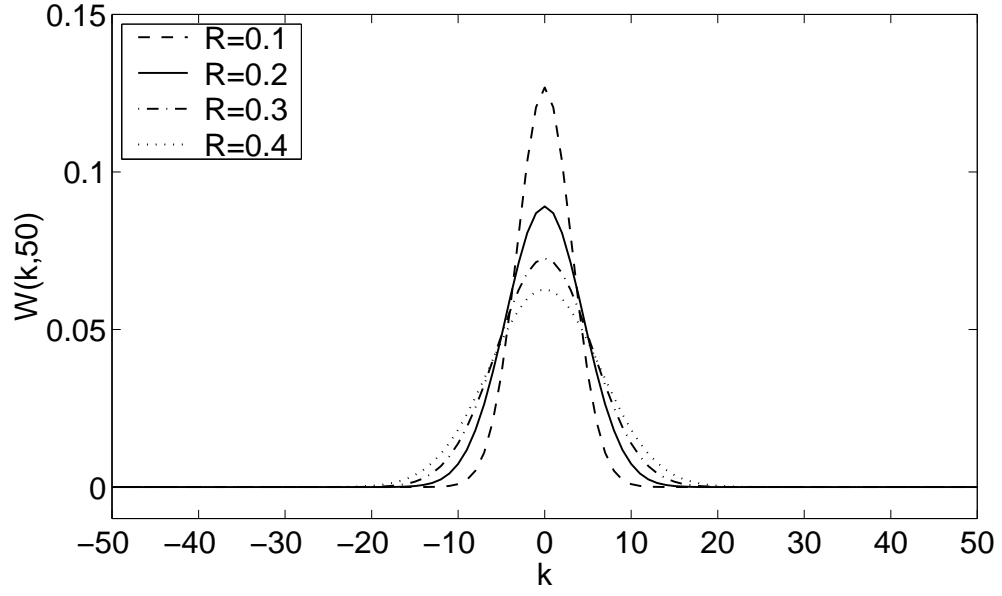| k		$R = 0.4$	$R = 0.1$
0	$924R^6 - 1512R^5 + 1050R^4 - 400R^3 + 90R^2 - 12R + 1$	0.181824	0.390804
1	$-792R^6 + 1260R^5 - 840R^4 + 300R^3 - 60R^2 + 6R$	0.154368	0.227808
2	$495R^6 - 720R^5 + 420R^4 - 120R^3 + 15R^2$	0.12672	0.065295
3	$-220R^6 + 270R^5 - 120R^4 + 20R^3$	0.07168	0.01048
4	$66R^6 - 60R^5 + 15R^4$	0.039936	0.000966
5	$-12R^6 + 6R^5$	0.012288	0.000048
6	R^6	0.004096	0.000001
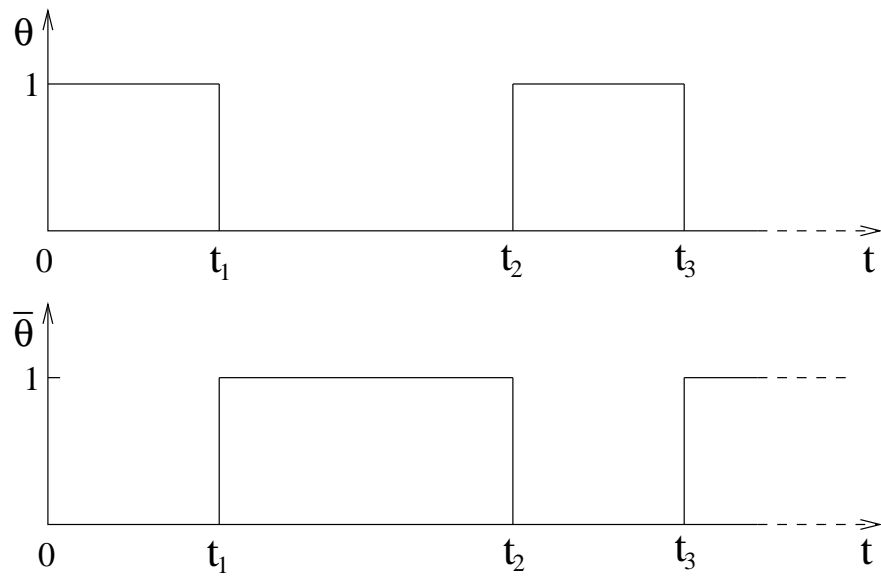
FIG. 1. The control functions.

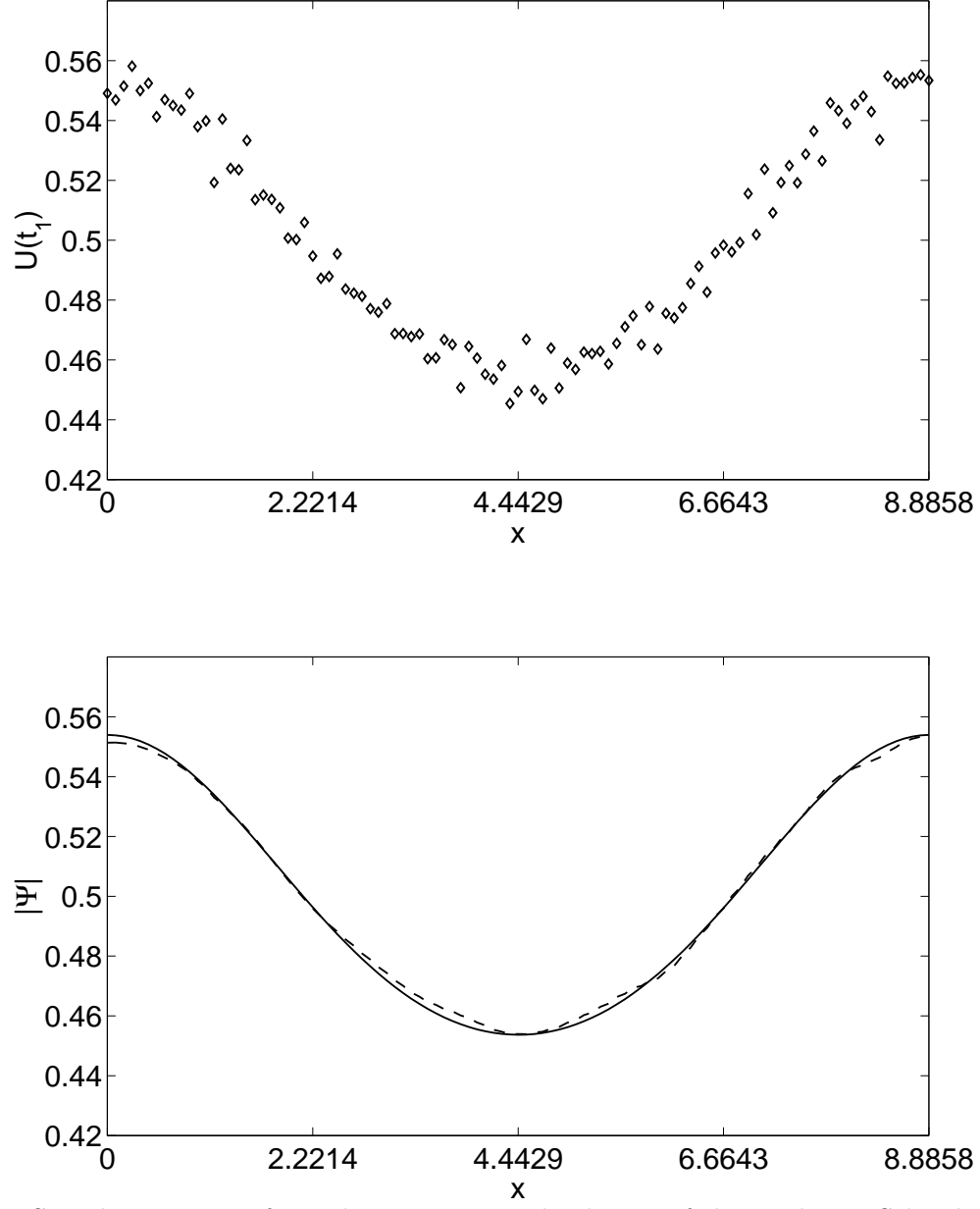FIG. 2. The filter weights of the controlled coupling process ($T = 0$).

FIG. 3. Signal restoration from the contaminated solution of the nonlinear Schrödinger equation. Upper: the contaminated solution $U(t_1)$; Lower: the restored solution (the dashed line) and the noise-free solution (the solid line).
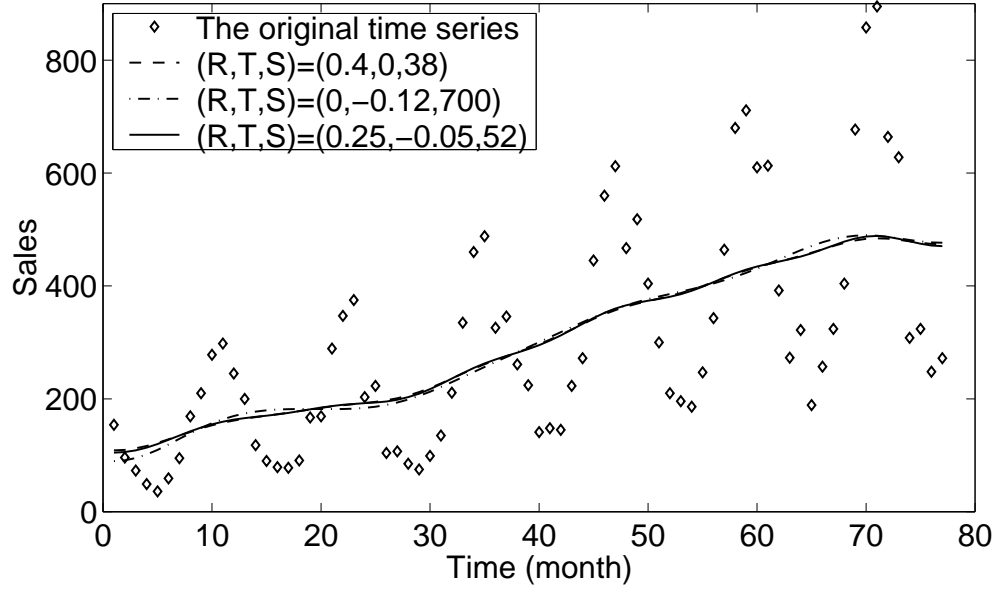
FIG. 4. Trend estimation by using the controlled coupling process.